Exercise 2

1. Find the distribution functions corresponding to the following density functions:

(i)	$f_X(x) = 1/[\pi(1+x^2)]$	$-\infty < x < \infty$	(Cauchy)
(ii)	$f_X(x) = e^{-x}/(1+e^{-x})^2$	$-\infty < x < \infty$	(Logistic)
(iii)	$f_X(x) = (a-1)/(1+x)^a$	$0 < x < \infty$	(Pareto)
(iv)	$f_X(x) = c\tau x^{\tau-1}e^{-cx^\tau}$	$0 < x < \infty$, $\tau > 0$, $c > 0$	(Weibull).

2. Find (without generating functions) the mean and the variance for the following distributions

   (a) $f_X(x) = \begin{cases} e^{-kx}x^{r-1}k^r/(r-1)! & x \geq 0 \\ 0 & x < 0 \end{cases}$ $\quad r > 0$, $k > 0$

   (The Gamma Distribution)

   (b) $f_X(x) = e^{-\lambda}\lambda^x/x!$ $\quad x = 0, 1, 2, \ldots.$ $\lambda > 0$

   (Poisson Distribution)

   (c) $f_X(x) = \binom{a+x-1}{x}\left[\frac{b}{1+b}\right]^a\left[\frac{1}{1+b}\right]^x$ $\quad x = 0, 1, 2, \ldots.$ $a > 0$, $b > 0$

   (Negative Binomial)

   (d) $f_X(x) = \frac{a-1}{(1+x)^a}$ $\quad x > 0$, $a > 1$

   (Pareto)

3. Show that if a random variable X is normally distributed with density, $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$ for $-\infty < x < \infty$, then $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

4. Suppose that X is a continuous random variable taking values between $-\infty$ and ∞ with distribution function $F(x)$. Sometimes we want to *fold* the distribution of X about the value $x = a$, that is we want the distribution function $F_Y(y)$ of the random variable Y obtained from X by taking $Y = X - a$ if $X > a$ and $Y = a - X$ if $X < a$. Find $F_Y(y)$ by working out directly $P(Y \leq y)$. What is the density function of Y? A particularly important application is the case when X has a $N(\mu, \sigma^2)$ distribution, and $a = \mu$. Apply your result to this case.

5. Show that if X is a continuous positive random variable

$$E[X] = \int_0^\infty [1 - F_X(x)]dx\ .$$

   (Try integration by parts.)

6. If X is a continuous positive continuous variable with density function $f_X(x)$ and mean μ, show that

$$f(y) = \begin{cases} 0 & y < 0 \\ yf_X(y)/\mu & y \geq 0 \end{cases}$$

   is a density function, and hence show that

$$\mathrm{E}(X^3)\mathrm{E}(X) \geq \{\mathrm{E}(X^2)\}^2.$$

7. A random variable has 'no memory' if for all x and for $y > 0$

$$P[X > x + y \mid X > x] = P[X > y].$$

Show that if X has either the exponential distribution, or a geometric distribution with $\text{Prob}(X = x) = q^{x-1}p$, then X has no memory. Interpret this property.

8. In a simple queueing system, the time spent in the queue has the following distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - \rho \exp(-\lambda(1-\rho)x) & x \geq 0 \end{cases}$$

where $\lambda \geq 0$ and $0 < \rho < 1$. Check that this is a distribution function, and describe the form of the distribution in words.

9. If X is a nonnegative continuous random variable, then

$$\bar{F}(x) = P(X > x)$$

is called the *survival function*, and

$$f(x)/\bar{F}(x)$$

is called the *failure rate* at x.

(a) Find the failure rate for the Weibull and Pareto distributions.

(b) Show that in general the failure rate does not decrease as x increases if

$$\bar{F}(x + y)/\bar{F}(x)$$

does not increase as x increases, for all $y \geq 0$. (Differentiate the logarithm of the given expression with respect to x.)